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The conservation equations for a non-equilibrium plasma

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The paper is concerned with formulation of the gas dynamic conservation equations for the individual species in a non-equilibrium partially ionized gas mixture. As an example, the conservation equations for the electrons and the overall conservation equations are developed for a three component plasma consisting of electrons, singly-ionized positive ions and neutral atoms. Non-elastic collisions are represented by the *collisional-radiative decay* mechanism of Bates, Kingston & McWhirter (1962*a*, *b*). Maxwellian velocity distributions are assumed, but the electrons are allowed to have a temperature different from the heavier particles and to drift relative to them. Particular attention is given to the electron energy balance equation which differs from that used by other investigators.

1. Introduction

One of the major problems in the design of magnetogasdynamic devices is to obtain a high electrical conductivity while, at the same time, keeping the gas temperature as low as possible. Many authors have suggested that non-equilibrium phenomena may be of use in solving this problem. One possible approach is to seek conditions under which the electron temperature can be maintained above the ambient gas temperature. Since the electrical conductivity increases with the electron temperature, this can produce a useful gain in conductivity for a given gas temperature. Various techniques for producing high electron temperatures have been discussed in the literature involving the use of applied or induced electro-magnetic fields (e.g. Kerrebrock 1961; Nue 1962), or rapid expansion of the ionized gas (e.g. McNab & Lindley 1962).

Non-equilibrium electron temperatures may also be of interest in purely gasdynamic problems involving ionized gases. Examples include the flow of an ionized gas through shock waves (e.g. Grewal & Talbot 1963), and expansion nozzles (e.g. Bray 1963).

It is possible to consider a separate electron temperature in these situations because, as a consequence of the large ratio of atom or ion mass to electron mass, electrons transfer energy rapidly in collisions with other electrons but only slowly in collisions with atoms or ions. This phenomenon is most pronounced in plasmas which do not contain molecules with internal degrees of freedom. The electron

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temperature must be determined from a macroscopic energy balance equation for the electron gas which should include all significant energy sources and losses. Previous authors have employed many different approximations to the electron energy balance equation, some of these are mentioned briefly in § 7 below. The main objective of the present work is to obtain an electron energy balance equation in conjunction with the other conservation equations, which are all consistent with certain stated assumptions.

This energy equation can in principle be obtained by the rigorous methods of the kinetic theory of gases (e.g. Kaufman 1960). However, such methods become very complex when, as in the present case, it is necessary to include the effects of inelastic collisions and long-range Coulomb interactions in addition to the effects of fluid motion. Here we adopt a simpler approach starting from a general balance equation for a small macroscopic element. This equation contains terms representing the interaction between a typical particle and all the other particles in the gas mixture; these interaction terms are later evaluated for a simple model.

2. The conservation of mass, momentum and energy

Consider a gas mixture made up of various species, s, and let the number density of particles of type s be $n_s(\mathbf{x}, t)$, where $\mathbf{x}$ is a position vector and t denotes time. A general macroscopic balance equation for a property of species s may be written

$$\frac{\partial}{\partial t}(n_s\langle\phi_s\rangle)+\frac{\partial}{\partial x^j}(n_s\langle\phi_s v_s^j\rangle) = I(\phi_s), \tag{1}$$

where $\langle\phi_s\rangle$ is the value of a property ϕ_s for a particle averaged over a macroscopically small volume element $\Delta\tau$, and $\langle\phi_s v_s^j\rangle$ is the mean value of ϕ_s multiplied by the component of its velocity in the direction x^j. The source term $\Delta\tau I(\phi_s)$ represents the changes in $\langle\phi_s\rangle$ which are caused by external agencies, such as gravity or an applied electro-magnetic field, plus the changes brought about by the interaction of all the particles present throughout the entire gas mixture on the s-type particles contained within $\Delta\tau$. This term includes the mutual interaction among the s-type particles themselves.

If we identify the property ϕ_s with the particle mass m_s, then the source term in (1) is simply the mass rate of production of species s, per unit volume, denoted by $\dot{w}_s$. The mass conservation equation is then given by

$$\frac{\partial\rho_s}{\partial t}+\frac{\partial}{\partial x^j}(\rho_s u_s^j) = \dot{w}_s, \tag{2}$$

where $\rho_s = m_s n_s$, is the mass density and $u_s^j = \langle v_s^j\rangle$, is the mean flow velocity of the species. Similarly, identifying the property $\langle\phi_s\rangle$ with $\langle m_s v_s^i\rangle$, the sth component of the momentum balance equation is

$$\frac{\partial}{\partial t}(\rho_s u_s^i)+\frac{\partial}{\partial x^j}(\rho_s u_s^i u_s^j)+\frac{\partial P_s^{ij}}{\partial x^j} = I(m_s v_s^i), \tag{3}$$

where $P_s^{ij} = \rho_s\langle c_s^i c_s^j\rangle$ is the partial pressure tensor, $c_{qs}^i = v_{qs}^i - u_s^i$ is the non-directed component of the velocity of a particular particle q of species s and, as before, the notation $\langle\ \rangle$ implies an average over the particles in the small element

$\Delta\tau$. The term $I\langle m_s v_s^i\rangle$ is the rate of change per unit volume of the ith component of momentum due to encounters between particles and externally applied forces. We shall break this term down by writing

$$I(m_s v_s^i) = \frac{1}{\Delta\tau}\sum_q\sum_k F_{qsk}^i + \frac{1}{\Delta\tau}\sum_q G_{qs}^i + C_s^i, \tag{4}$$

where F_{qsk}^i is the time averaged force exerted by the entire species k, during elastic encounters, on the qth particle of species s contained within the small volume element. The q-summation extends over all the s-type particles within $\Delta\tau$ and the summation over the species includes s itself. The quantity G_{qs}^i is the ith component of the externally applied force on a particle q, and C_s^i is the rate at which momentum is produced per unit volume as a result of non-elastic encounters.

When the property ϕ_{qs} is set equal to the total energy of the qth particle, i.e.

$$\phi_{qs} = \psi_{qs} + \tfrac{1}{2}m_s v_{qs}^i v_{qs}^i,$$

where ψ_{qs} is the internal energy of the particle, we obtain the species energy conservation equation

$$\frac{\partial}{\partial t}[\epsilon_s + \tfrac{1}{2}\rho_s u_s^i u_s^i] + \frac{\partial}{\partial x^j}[(\epsilon_s + \tfrac{1}{2}\rho_s u_s^i u_s^i)u_s^j + P_s^{ij}u_s^i + q_s^j] = I(\psi_s + \tfrac{1}{2}m_s v_s^i v_s^i), \tag{5}$$

where $$\epsilon_s = n_s\langle\psi_s\rangle + \tfrac{1}{2}\rho_s\langle c_s^i c_s^i\rangle$$

is the total internal energy of the species per unit volume and $(\partial q_s^j/\partial x^j)$ represents the thermal conduction within constituent s. The right-hand side in equation (5) represents the work done by the interaction and applied forces plus the rate at which energy is supplied to the species s as a consequence of the non-elastic encounters; thus we write

$$I(\psi_s + \tfrac{1}{2}m_s v_s^i v_s^i) = \frac{1}{\Delta\tau}\sum_q\sum_k F_{qsk}^i v_{qs}^i + \frac{1}{\Delta\tau}\sum_q\sum_k G_{qs}^i v_{qs}^i + Q_s, \tag{6}$$

where Q_s is the energy supplied per unit volume and time in non-elastic encounters. Combining (5) and (6) and eliminating the kinetic energy, $\tfrac{1}{2}\rho_s u_s^i u_s^i$, with the aid of (3) and (4), the energy balance equation may finally be written in the form

$$\frac{\partial\epsilon_s}{\partial t} + \frac{\partial}{\partial x^j}[\epsilon_s u_s^j + P_s^{ij}u_s^i] = u_s^i\frac{\partial P_s^{ij}}{\partial x^j} - \frac{\partial q_s^j}{\partial x^j} + Q_s + L_s + M_s, \tag{7}$$

where $$L_s = \frac{1}{\Delta\tau}\sum_q\sum_k F_{qsk}^i c_{qs}^i \tag{8}$$

and $$M_s = \tfrac{1}{2}u_s^i u_s^i \dot{w}_s - u_s^i C_s^i. \tag{9}$$

3. General equations for elastic encounters

We shall now be more specific about the composition of our gas mixture by assuming that some species have a net particle charge e_{qs}. The encounters between charged particles, in which both momentum and energy will be exchanged, can then occur when they are at relatively large distances apart as a consequence of their associated Coulomb fields. However, we know that a particular charged particle will be effectively shielded from the individual fields of other charged particles, situated at a distance greater than the 'Debye length' from it, by the collective motion of the oppositely charged particles in its immediate vicinity. The cumulative effect of the fields of all the charged particles outside the Debye sphere on the particular charged particle may not be insignificant and to account for this effect we shall assume that it may be represented by an induced electric field $\bar{E}^i$ and an induced magnetic field $\bar{B}^i$; these induced fields are macroscopic quantities, averaged over finite intervals of space and time. Clearly, $\bar{E}^i$ may be related to the space-wise distribution of the charge density in the gas mixture (plasma) as a whole and $\bar{B}^i$ to the mean motion of all the charged particles within the plasma. Therefore, we express the average force on a particular charged particle due to encounters with all the remaining charged particles situated outside the Debye sphere in the form

$$\sum_{\substack{k \\ (d>\lambda)}} F^i_{qsk} = e_{qs}[\bar{E}^i + (\mathbf{v}_{qs} \wedge \bar{\mathbf{B}})^i], \tag{10}$$

where λ is the Debye length. The fields $\bar{E}^i$ and $\bar{B}^i$ may be reinforced by an externally applied electromagnetic field $(\bar{\bar{E}}^i, \bar{\bar{B}}^i)$ which, in the absence of any other external agency, would be responsible for the applied force G^i_{qs}. The sum of these forces on all the s-type charged particles within our original elemental volume, $\Delta\tau$, is then

$$\sum_q \sum_{\substack{k \\ (d>\lambda)}} F^i_{qsk} + \sum_q G^i_{qs} = n_s e_s[E^i + (\mathbf{u}_s \wedge \mathbf{B})^i]\,\Delta\tau, \tag{11}$$

where $e_{qs} = e_s$, and the fields $E^i = \bar{E}^i + \bar{\bar{E}}^i$, and $B^i = \bar{B}^i + \bar{\bar{B}}^i$, are assumed to be continuous functions of position and time.

The rate at which energy is transferred to the charged species by long-range encounters is obtained by multiplying equation (10) by v^i_{qs} and then summing over the volume element; the result is simply

$$\sum_q \sum_{\substack{k \\ (d>\lambda)}} F^i_{qsk} v^i_{qs} + \sum_q G^i_{qs} v^i_{qs} = n_s e_s E^i u^i_s \Delta\tau, \tag{12}$$

where we have also taken into account the work done by the externally applied field.

Although it is clearly unrealistic to treat encounters between charged particles as independent binary collisions we shall, nevertheless, adopt this approach with the knowledge that the form of the resulting expressions for such gross plasma properties as electrical conductivity, are substantiated by experimental measurements (e.g. Lin, Restler & Kantrowitz 1955). These binary collision processes are more easily considered by introducing the normalized single species velocity distribution function of classical kinetic theory. This is defined such that the probable proportion of the total number of particles which, at time t, are situated in the volume element $\Delta\tau$ and have velocities lying in the range $\Delta\mathbf{v}_s$ centred about $\mathbf{x}$ and $\mathbf{v}_s$ respectively, is equal to

$$f_s(\mathbf{x}, \mathbf{v}_s, t)\,\Delta\mathbf{v}_s\,\Delta\tau.$$

In terms of the velocity distribution function, the average values of the molecular properties referred to in § 2 are

$$\langle\phi_s\rangle = \int \phi_s f_s d\mathbf{v}_s; \quad \langle\phi_s \mathbf{v}_s\rangle = \int \phi_s \mathbf{v}_s f_s d\mathbf{v}_s,$$

where the integration extends over the entire velocity space.

By considering single binary encounters between particles of different species the standard methods of kinetic theory yield the following results for the rate at which all the s-type particles contained within $\Delta\tau$ lose momentum (kinetic energy) as a consequence of encounters with particles of species k:

$$-\sum_{\substack{q \\ (d<h)}} \mathbf{F}_{qsk} = \frac{m_s m_k}{m_s + m_k} n_s n_k \int \mathbf{g}\sigma_{sk}(g)\, g f_k f_s d\mathbf{v}_k d\mathbf{v}_s \Delta\tau, \tag{13}$$

$$-\sum_{\substack{q \\ (d<h)}} \mathbf{F}_{qsk}\cdot\mathbf{v}_{qsk} = \frac{m_s m_k}{m_s + m_k} n_s n_k \int (\mathbf{g}\,.\,\mathbf{G})\, \sigma_{sk}(g)\, g f_k f_s d\mathbf{v}_k d\mathbf{v}_s \Delta\tau. \tag{14}$$

The vector quantity $\mathbf{g}$ is the relative velocity before the encounter, $\mathbf{G}$ is the velocity of the centre of mass of the two colliding particles, $\sigma_{sk}(g)$ is the scattering cross-section and h is the upper limit of the impact parameter for the two colliding particles. Clearly the expressions (13) and (14) can only be evaluated when the form of the velocity distribution functions and the appropriate scattering cross-sections are known.

4. The elastic interaction terms for a Maxwellian velocity distribution

Before we proceed to simplify and evaluate the elastic interaction terms, it will be convenient to be more specific about the composition of our plasma. We shall assume that it is composed of electrons (suffix e), singly ionized positive ions (suffix i) and neutral atoms (suffix a), all of which have spherically symmetric force fields.

If we consider the flow of our gas mixture allowing for only *small* gradients of the macroscopic properties it will correspond, in the absence of applied fields and interaction terms (i.e. $I(\phi_s) = 0$), to the limiting case of isentropic flow in which the velocity distribution functions are of Maxwellian form. Kantrowitz & Petschek (1957) have argued for the case of a simple plasma that, provided the ion Larmor radius is larger than the effective mean free path, the velocity distribution functions for both the electrons and the ions can still be closely approximated by a Maxwellian form over a wide range of conditions. However, equipartition between the translational energy of the heavy particles (ions and neutrals) and the electrons may not be complete since, as we see from the results of § 3, the energy exchanged in electron/ion encounters is directly proportional to the mass ratio $m_e{:}m_i$, and will therefore require many collisions for equipartition. In view of these comments we shall assume that the species velocity distribution functions have a Maxwellian form in a frame of reference moving with the mean flow velocity of the species, thus

$$f_s = \left(\frac{m_s}{2\pi k T_s}\right)^{\frac{3}{2}} \exp\left(-\frac{m_s c_s^2}{2kT_s}\right),$$

where k is Boltzmann's constant and $T_s = m_s\langle c_s^2\rangle/3k$ is the species temperature.

Recently, Morse (1963) has used the above assumption to evaluate the interaction terms for the general case of elastic binary encounters between specific types of particle, account being taken of the different species mean flow velocities, by taking the appropriate moments of the *spatially homogeneous* Boltzmann equation. The evaluation of the integrals is straightforward but tedious. However, since we are primarily interested in a simple plasma in which the ions and the neutrals have almost identical masses, much greater than the electron mass, a number of assumptions may be made which still allow us to consider a wide range of plasmas whilst simplifying the reduction of integrals (13) and (14).

We assume that:

(*a*) the mean thermal energy of the ions and neutrals is less than or of the same order of magnitude as that of the electrons, so we count

$$c_i = O[c_e(m_e/m_i)^{\frac{1}{2}}];$$

(*b*) the ions and the neutrals are at the same temperature and have the same mean flow velocity, $u_i^i = u_a^i = u^i$;

(*c*) the electron diffusion velocity $\omega_e^i = u_e^i - u^i$ satisfies the condition

$$\omega_e/\langle c_e^2\rangle^{\frac{1}{2}} = O[(m_e/m_i)^{\frac{1}{2}}].$$

The actual reduction of the integrals is straightforward but tedious. The resulting expressions take the form:

$$\mathbf{F}_{qia} = \mathbf{F}_{qai} = 0, \tag{15}$$

$$\sum_k \sum_{\substack{q \\ (d<h)}} \mathbf{F}_{qek} = -n_e m_e \sum_k \nu_{ek} \boldsymbol{\omega}_e \Delta\tau, \tag{16}$$

$$\sum_k \sum_{\substack{q \\ (d<h)}} \mathbf{F}_{qek} \cdot \mathbf{v}_{qe} = 2n_e m_e \sum_k \frac{\nu_{ek}}{m_k} [\tfrac{3}{2}k(T_k - T_e) - \tfrac{1}{2}m_k \boldsymbol{\omega}_e \cdot \mathbf{u}]\,\Delta\tau, \tag{17}$$

where ν_{ek} is the effective collision frequency of the electrons with species k; it may be evaluated using the expression

$$\nu_{ek} = \frac{4\pi n_k m_e}{3kT_e} \int f_e \sigma_{ek}(c_e)\, c_e^5\, dc_e.$$

Since $\sum_k \sum_{\substack{q \\ (d>h)}} \mathbf{F}_{qek} \cdot \mathbf{c}_{qe}$ is zero, the quantity L_e defined in (8) is

$$L_e = 2n_e m_e \sum_k \frac{\nu_{ek}}{m_k} [\tfrac{3}{2}k(T_k - T_e) + \tfrac{1}{2}m_k \boldsymbol{\omega}_e^2]. \tag{18}$$

For the important case of Coulomb collisions between electrons and ions

$$\sigma_{ei} = 2\pi \frac{e^4}{m_e^2 g^4} \ln \left[1 + \frac{h^2 m_e^2 g^4}{e^4}\right],$$

where e is the magnitude of the electronic charge measured in esu and h is the upper limit of the impact parameter which is usually taken to be the Debye shielding length, $\lambda = [kT_e/4\pi n_e e^2]^{\frac{1}{2}}$. The electron/ion collision frequency is given by the expression

$$\nu_{ei} = \frac{8}{3}\left(\frac{\pi}{m_e}\right)^{\frac{1}{2}} n_i e^4 \frac{1}{(2kT_e)^{\frac{3}{2}}} \ln \left[\frac{k^3 T_e^3}{\pi n_e e^6}\right], \tag{19}$$

a result which is in close agreement with that obtained by Petschek & Byron (1957), see also Goldsworthy (1961), where no account was taken of the finite value of the electron diffusion velocity. As we shall see later, the electrical conductivity for a fully ionized plasma in the absence of a magnetic field is then given by $\eta_{ei}^{-1} = n_e e^2/m_e \nu_{ei}$, which differs from the Spitzer (1956) conductivity by only a small numerical factor ($\doteqdot 3{\cdot}4$).

Because of the relatively complicated way in which the electrons are scattered by the atomic field of some neutral atoms, the Ramsauer–Townsend effect, see Massey & Burhop (1952), it may be useful to assume an effective scattering cross-section which is taken to be constant over the likely range of electron temperature. In this case the electron/neutral collision frequency is given by

$$\nu_{ea} = n_a \sigma_{ea} \left(\frac{8kT_e}{\pi m_e}\right)^{\frac{1}{2}}, \tag{20}$$

which, when applied to the case of a slightly ionized gas, gives a value for the electrical conductivity only 12% less than that given by Chapman & Cowling (1960).

There is no conceptual difficulty in taking into account finite values of *all* the relative diffusion velocities, as has been demonstrated by Morse. However, for encounters between particles of similar mass, the relative diffusion velocity must be much less than the mean thermal velocities if Maxwellian velocity distributions are to be employed.

5. Non-elastic encounters

We shall continue to consider a simple plasma model and represent the non-elastic encounters by the *collisional-radiative decay* mechanism proposed by Bates *et al.* (1962*a*, *b*) for a partially ionized hydrogenic plasma. The reactions considered are: three-body recombination and ionization

$$H^+ + e^- + e^- \underset{}{\overset{K(e^-,\,p)}{\rightleftharpoons}} H(p) + e^-,$$

superelastic and inelastic collisions

$$H(p) + e^- \overset{K(p,\,q)}{\rightleftharpoons} H(q) + e^- \quad (p > q),$$

transition by line radiation and absorption

$$H(p) \overset{A(p,\,q)}{\rightleftharpoons} H(q) + h\nu \quad (p > q),$$

radiative recombination and photo ionization

$$H^+ + e^- \overset{\beta(p)}{\rightleftharpoons} H(p) + h\nu,$$

where $p, q, \ldots$, are the principal quantum numbers of discrete energy levels and e^- refers to an electron in the free state. We may regard $K(e^-, p)$, $K(p, q)$ etc. as phenomenological rate coefficients for the various reactions which are identical to those defined by Bates *et al.* (1962*a*). The rates at which the number densities of the excited state atoms change as a consequence of the above reactions may be written

$$\dot{n}_a(p) = K(e^-,p)\,n_e^2 - K(p,e^-)\,n_a(p)\,n_e - \sum_{q=1}^{\infty} K(p,q)\,n_a(p)\,n_e + \sum_{q=1}^{\infty} K(q,p)\,n_a(q)\,n_e + \sum_{q=p}^{\infty} A(q,p)\,n_a(q) - \sum_{q=1}^{p} A(p,q)\,n_a(p) + \beta(p)\,n_e^2, \quad (21)$$

where we have assumed the plasma to be *optically thin*, i.e. all emitted radiation escapes. Bates *et al.* argue that, for an optically thin plasma, if the mean thermal energy of the electrons is much less than the first excitation energy and

$$n_e \ngtr 10^{14+w}\,\text{cm}^{-3},$$

where $2^w = T_e \times 10^{-3}$, quasi-equilibrium number densities of the excited systems are very rapidly established which change very little by comparison with the number densities of the free electrons, n_e, and ground state atoms, $n_a(1)$. Thus, the *total* mass rate of production of the neutrals atoms, $\dot{w}_a$ in equation (2), is simply written

$$\dot{w}_a/m_a = -\dot{w}_i/m_i = -\dot{w}_e/m_e = \dot{n}_a(1), \quad (22)$$

since $\dot{n}_i = \dot{n}_e = -\dot{n}_a(1)$, and $m_a = m_i + m_e$.

From the set of linear equations (21), excluding the equation for $\dot{n}_a(1)$, values of $n_a(p)$ may be calculated in terms of $n_a(1)$, n_e and the rate coefficients by assuming $\dot{n}_a(p) = 0$, for $p > 1$. An infinite matrix may be avoided by taking advantage of the fact that when p is large enough the collisional processes are much more important than the radiative processes so $n_a(p)$ is given by the Saha equation

$$\frac{n_a(p)}{n_e^2} = p^2\left(\frac{h^2}{2\pi m_e kT_e}\right)^{\frac{3}{2}} \exp\left(\frac{I_p}{kT_e}\right) \quad \text{(for large } p\text{)},$$

where I_p is the ionization energy from level p and h is Plank's constant. Thus, $\dot{n}_a(1)$ may be evaluated from equation (21), after substitution, by specifying the electron number density and the rate coefficients; these latter quantities being functions of the electron temperature. Bates *et al.* have carried out calculations to determine $\dot{n}_a(1)$ for a wide range of electron density and temperature and have expressed their results by tabulating the quantities α and S where

$$\dot{n}_a(1) = -\dot{n}_i = \alpha n_i n_e - S n_a(1)\,n_e.$$

The quantity α was given the name *collisional-radiative recombination coefficient* and S the name *collisional-radiative ionization coefficient.*

Recent investigations by Hinnov & Hirschberg (1962), Byron, Stabler & Bortz (1962) and Makin & Keck (1963) have resulted in relatively simple expressions for the rate coefficients which appear to be valid over certain ranges of conditions. For instance, in the important range where radiative recombination and radiative transitions may be neglected (i.e. high electron density and low temperature), Makin & Keck have developed a classical variational theory of three-body electron-ion recombination giving α in the form

$$\alpha = 2{\cdot}3 \times 10^{-8}\,T_e^{-\frac{9}{2}}\,n_e\,\text{cm}^3\,\text{sec}^{-1},$$

which is in close agreement with the numerical results of Bates *et al.* The ionization coefficient follows directly from the equilibrium relationship

$$\frac{\dot{\alpha}}{Sn_e} = \left(\frac{h^2}{2\pi m_e kT_e}\right)^{\frac{3}{2}} \exp\left(\frac{I_1}{kT_e}\right).$$

In a sufficiently tenuous plasma, radiative recombination is the dominant process and α is given by the approximate expression

$$\alpha \doteqdot 4\cdot 1 \times 10^{-10} T_e^{-\frac{3}{4}}\,\mathrm{cm}^3\,\mathrm{sec}^{-1}.$$

Bates's numerical results at very low electron densities are correlated reasonably well by the above expression.

Due to the smallness of the electron mass we shall neglect the inertial terms, $\partial(\rho_e u_e^i)/\partial t$ and $\partial(\rho_e u_e^i u_e^j)/\partial x^j$, in the electron momentum equation. For the same reason we shall also neglect the electron momentum source term, $C_e^i \Delta\tau$, in equation (4) and the term M_e in equation (7). It is then logical to assume that the ion and neutral momentum source terms are given by

$$C_i^i = -C_a^i = -\dot{n}_a(1)\, m_a u^i \quad (m_i \doteqdot m_a),$$

since this at least satisfies the necessary condition $\sum\limits_s C_s^i = 0$, and no theory is available which takes into account a mean relative velocity between species when considering non-elastic encounters.

We shall now proceed to deduce the electron energy source term, Q_e, by continuing to consider the collisional-radiative decay mechanism as applied to an optically thin plasma. The net energy gained by the electrons per event of the three-body recombination process is I_p; the net loss of energy by the electrons per event of the collisional ionization process is also I_p. The energy gained (lost) by the electrons in one event of the superelastic (inelastic) collisions is clearly $(I_p - I_q)$. Finally, of the total energy radiated away from the plasma during radiative recombination, the electrons lose the average amount of energy $(3kT_e/2)$, per event. Thus we write

$$Q_e = \sum_{p=1}^{\infty} K(e^-,p)\, n_i n_e I_p - \sum_{p=1}^{\infty} K(p,e^-)\, n_a(p)\, n_e I_p + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} K(p,q)\, n_a(p)\, n_e (I_p - I_q)$$
$$- \sum_{p=1}^{\infty}\sum_{q=p}^{\infty} K(p,q)\, n_a(p)\, n_e (I_q - I_p) - \sum_{p=1}^{\infty} \beta(p)\, n_i n_e (3kT_e/2). \quad (23)$$

By multiplying equation (21) by I_p and then summing over all p an expression is obtained which enables us to replace all the collisional terms in equation (23) (terms containing the coefficients $K(e^-,p)$, $K(p,e^-)$ and $K(p,q)$) by radiation terms and terms involving $\dot{n}_a(p)$, viz.

$$Q_e = \sum_{p=1}^{\infty} \dot{n}_a(p)\, I_p + Q_{\mathrm{rad.}}, \quad (24)$$

where $$Q_{\mathrm{rad.}} = -\sum_{p=1}^{\infty}\sum_{q=1}^{p} A(p,q)\, n_a(p)\,(I_p - I_q) - \sum_{p=1}^{\infty} \beta(p)\, n_e^2 (I_p + 3kT_e/2). \quad (25)$$

However, in an optically thin plasma, $\dot{n}_a(p) = 0$ for $p > 1$, therefore

$$Q_e = \dot{n}_a(1)\, I_1 + Q_{\mathrm{rad.}}. \quad (26)$$

The above equation simply states that the rate at which electrons gain energy by the *collisional-radiative decay* mechanism is equal to the rate at which energy is liberated in forming ground state atoms from free electrons and ions, *less* the rate at which energy is radiated away from the plasma as a whole.

If the plasma is optically thick for particular radiation then the appropriate terms are omitted from the expression for $Q_{\text{rad.}}$. For instance, if Lyman line radiation is absorbed, then all terms containing the transition probabilities $A(p, 1)$ are omitted from equation (25). In this particular example the level 2 would be effectively stabilized with respect to the radiative transitions and it may be necessary to retain the term, $\dot{n}_a(2)\, I_2$, in equation (24). This and other examples of optically thick plasmas are discussed by Bates *et al.* (1962*b*).

The only other energy loss mechanism which may need to be accounted for in our non-relativistic electron energy balance is bremsstrahlung radiation. However, it is unlikely that this will be important other than at the very high temperatures where our hydrogenic model may be fully ionized and hence Q_e would be zero.

The average amount of energy lost by the ion gas per recombination event is

$$(\tfrac{3}{2}kT_i + \tfrac{1}{2}m_i u^2),$$

which, in turn, is equal to the energy gained by the neutral gas atoms. The ions gain this amount of energy at the expense of the neutral gas per ionization event. We are, of course, assuming that during the average recombination/ionization event there is no significant change in the translational energy of the heavy particle, therefore

$$Q_i = \dot{n}_i(\tfrac{3}{2}kT_i + \tfrac{1}{2}m_i u^2)$$

and

$$Q_a = \sum_{p=1}^{\infty} \dot{n}_a(p)\,(\tfrac{3}{2}kT_a + \tfrac{1}{2}m_a u^2),$$

where $T_i = T_a$, and $m_i = m_a$, by our earlier assumptions. In the case of the optically thin plasma

$$Q_a = \dot{n}_a(1)\,(\tfrac{3}{2}kT_a + \tfrac{1}{2}m_a u^2).$$

In applying the theory of Bates *et al.* to non-elastic processes occurring in a non-uniform plasma, we have implicitly assumed that deviations from Maxwellian velocity distributions and the macroscopic drift velocities of charged particles have only a small effect on the non-elastic processes.

6. Final form of the conservation equations assuming Maxwellian velocity distributions

Because of the assumed Maxwellian velocity distributions, the partial pressure tensors, P_s^{ij}, become scalar quantities, p_s, and the thermal conduction terms, $\partial q_s^j/\partial x^j$, are zero. Also, since the electron mass is small, the conservation equations (2), (3) and (7) may be written for the electrons in the form:

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x^j}(n_e u_e^j) = \frac{\dot{w}_e}{m_e}, \tag{27}$$

$$\frac{\partial p_e}{\partial x^i} = -n_e e[E^i + (\mathbf{u}_e \wedge \mathbf{B})^i] + R_e^i, \tag{28}$$

$$\frac{\partial \epsilon_e}{\partial t} + \frac{\partial}{\partial x^j}[(\epsilon_e + p_e)\, u_e^j] = u_e^j \frac{\partial p_e}{\partial x^j} + Q_e + L_e, \tag{29}$$

where, from equations (22), (15), (18) and (24), respectively, the interaction terms may be written as

$$\dot{w}_e/m_e = -\dot{n}_a(1), \quad R_e^i = n_e m_e(\nu_{ea}+\nu_{ei})(u^i-u_e^i),$$

$$L_e = 2n_e(m_e/m_a)(\nu_{ea}+\nu_{ei})[\tfrac{3}{2}k(T-T_e)+\tfrac{1}{2}m_a(u^i-u_e^i)^2],$$

$$Q_e = \dot{n}_a(1)\,I_1 + Q_{\text{rad.}},$$

the last being for an optically thin plasma. The collision frequencies ν_{ei} and ν_{ea} may be evaluated from equations (19) and (20), respectively.

Overall conservation equations for the plasma are obtained by summing (2), (3) and (7) for all constituents and noting that $u^i = u_a^i = u_i^i$; the following results are obtained:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^j}(\rho u^j) = 0, \tag{30}$$

$$\rho\frac{Du^i}{Dt} = -\frac{\partial p}{\partial x^i} + (n_i - n_e)\,e[E^i + (\mathbf{u}\wedge\mathbf{B})^i] + n_e e[(\mathbf{u}-\mathbf{u}_e)\wedge\mathbf{B}]^i, \tag{31}$$

$$\rho\frac{DH}{Dt} - \frac{Dp}{Dt} = -n_e e[(\mathbf{u}-\mathbf{u}_e)\wedge\mathbf{B}]^i u^i + n_e e E^i(u^i - u_e^i) + \frac{\partial}{\partial x^j}[(\epsilon_e + p_e)(u^i - u_e^i)] + Q_e, \tag{32}$$

where the overall plasma properties are defined as

$$\rho = \sum_s \rho_s, \quad p = \sum_s p_s, \quad \epsilon = \sum_s \epsilon_s$$

and the quantity H is a 'frozen' specific enthalpy for the plasma, $H = (\epsilon + p)/\rho$; it contains no contribution from the ionization energy.

The above equations may be written in terms of the conduction current density

$$j^i = n_e e(u^i - u_e^i),$$

and the total electrical resistivity

$$\eta = \eta_{ei} + \eta_{ea}.$$

Using these relations, equation (28) reduces to a simple form of the generalized Ohm's law:

$$\eta j^i = E^i + (\mathbf{u}\wedge\mathbf{B})^i - \frac{1}{n_e e}\left[(\mathbf{j}\wedge\mathbf{B})^i - \frac{\partial p_e}{\partial x^i}\right] \tag{33}$$

and the overall energy equation becomes

$$\rho\frac{DH}{Dt} - \frac{Dp}{Dt} = j^i[E^i + (\mathbf{u}\wedge\mathbf{B})^i] + \frac{\partial}{\partial x^i}[(\epsilon_e + p_e)(u^i - u_e^i)] + Q_e. \tag{34}$$

In the latter equation, the first term on the right-hand side is the work performed by the electromagnetic field in a frame of reference moving with the plasma, and the second term represents transport of enthalpy due to the drift of the electrons relative to the mean motion of the plasma. The final term represents heat sources due to recombination and radiation phenomena. In many cases of practical importance it is the steady, time independent electron energy equation which is of interest, this takes the form

$$u_e^i n_e \frac{\partial}{\partial x^i} (\tfrac{5}{2}kT_e) = u_e^i \frac{\partial p_e}{\partial x^i} + j^2\eta + 2n_e \frac{m_e}{m_a} (\nu_{ei}+\nu_{ea}) \tfrac{3}{2}k(T-T_e) + (\tfrac{5}{2}kT_e + I_1)\, \dot{n}_a(1) + Q_{\text{rad.}}, \quad (35)$$

where the electron enthalpy per unit volume, $(\epsilon_e + p_e)$, has been written as $\frac{5}{2}n_e kT_e$.

7. Concluding remarks

The key assumptions in the analysis leading to the equations of the previous section are:

(1) the plasma is made up of three simple components—electrons, neutral atoms and singly ionized positive ions;

(2) the non-elastic processes are limited to the *collisional-radiative decay* mechanism, Bates *et al.* (1962*a*, *b*), with the elastic processes described by the classical theory for binary encounters;

(3) the only diffusion allowed is that of the electrons relative to the ions and neutrals with the restriction that the magnitude of the diffusion velocity is of the same order or less than $(m_e/m_i)^{\frac{1}{2}} \langle c_e^2 \rangle^{\frac{1}{2}}$;

(4) the distribution of each species throughout the particle velocity space is described by Maxwellian distribution functions and this eliminates all transport phenomena except those 'driven' by the electromagnetic field.

In spite of the crudity of this model for the flow of a non-equilibrium plasma, many real effects are predicted. No assumption has been made about charge neutrality and indeed equation (33) shows that, in the absence of the conduction current, an induced electric field is built up by charge separation. The magnitude of this field is predicted to be

$$\bar{E}^i = -(n_e e)^{-1} \partial p_e / \partial x^i.$$

There are many cases where the induced electric field is not negligible although numerically $n_e = n_i$, for example, the sheath effect around a probe in a plasma, Petschek & Byron (1957). It is a vestige of all the coupled thermo-electric effects which occur when diffusion is correctly represented.

A more sophisticated analysis would be required to eliminate the assumption of Maxwellian velocity distributions. However, examination of the general energy balance equation (7) applied to electrons does suggest that it will retain the same form as equation (29), in the presence of small deviations from Maxwellian equilibrium, with the addition of transport and dissipative terms. In many cases the most important of these neglected terms is likely to be the one representing thermal conduction within the electron gas.

Finally, the electron energy balance equation may be compared with previously published equations. Russell, Byron & Bortz (1963) quote without derivation an equation very similar to equation (35) but omitting the term $u_e^i\, \partial p_e / \partial x^i$. Many other authors, including Kerrebrock (1961), Brocher (1962) and McNab & Lindley (1962), have assumed simple, intuitively based energy balance equations in which the change of electron internal energy is balanced against heat sources such as Joule heating and electric collisions with other constituents. These authors also neglect the pressure gradient term. On the other hand, Kaufman (1960), in a detailed kinetic theory derivation, includes rigorously the effects of pressure gradients and transport phenomena but does not consider non-elastic collisions.

Grewal & Talbot (1963), quoting Kaufman, use an equation similar to our equation (29) in their studies of shock-wave structure. They include the additional terms representing thermal conduction and viscous dissipation but omit the inelastic collision terms and the terms containing the electrical conductivity. Their results suggest that thermal conduction in the electron gas is significant in their case. In such circumstances, we can add the thermal conduction and other dissipative terms to equation (29) in an *ad hoc* manner but then, of course, the equation is no longer strictly compatible with the initial assumptions.

Another situation where thermal conduction is likely to have a large effect in the rapid expansion of an ionized gas through a supersonic nozzle at low pressure. The experimental results of Clayden & Coleman (1963) show that high electron temperatures occur at the exit to such a nozzle. On the other hand, equation (35) would predict that, under experimental conditions where the recombination term $(\frac{5}{2}kT_e + I_1)\,\dot{n}_a(1)$ is small because of the low pressure and the Ohmic heating term $j^2\eta$ is small because of the absence of applied fields, T_e would fall in roughly the same way as T. Order of magnitude calculations using the thermal conductivity expression of Spitzer (1956) suggest that the neglected thermal conduction term could be large enough to explain the discrepancy in this case.

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REFERENCES

Bates, D. R., Kingston, A. E. & McWhirter, R. W. P. 1962*a* *Proc. Roy. Soc.* A, **267**, 297.

Bates, D. R., Kingston, A. E. & McWhirter, R. W. P. 1962*b* *Proc. Roy. Soc.* A, **270**, 155.

Bray, K. N. C. 1963 Article in *High Temperature Aspects of Hypersonic Flow*, pp. 67–87. London: Pergamon.

Brocher, E. F. 1962 *Symposium on magnetoplasmadynamic electrical power generation*. The Institution of Electrical Engineers, North Eastern Section. Session IV, Paper 8.

Byron, S., Stabler, R. C. & Bortz, P. I. 1962 *Phys. Rev. Letters*, **8**, 376.

Chapman, S. & Cowling, T. G. 1960 *The Mathematical Theory of Non-Uniform Gases*. Cambridge University Press.

Clayden, W. A. & Coleman, P. L. 1963 Distribution of electron density and temperature in an arc heated low density wind tunnel. *R.A.R.D.E. Memo* (B), 57/63, ARC. 25, 254.

Goldsworthy, F. A. 1961 Article in *Progress in Aeronautical Sciences*, **1**, p. 188. London: Pergamon.

Grewal, M. S. & Talbot, L. 1963 *J. Fluid Mech.* **16**, 4.

Hinnov, E. & Hirschberg, J. G. 1962 *Phys. Rev.* **125**, 795.

KANTROWITZ, A. R. & PETSCHEK, H. E. 1957 Article in *Magnetohydrodynamics*, pp. 3–15. Stanford University Press.

KAUFMAN, A. 1960 Article in *La Theorie des gaz Neutres et Ionises* (ed. de Witt & Detoeuf). New York: Wiley.

KERREBROCK, J. L. 1961 *Proc. 2nd Symp. on Engineering Aspects of Magnetohydrodynamics*, University of Pennsylvania.

LIN, S. C., RESTLER, E. L. & KANTROWITZ, A. R. 1955 *J. Appl. Phys.* **26**, 95.

McNAB, I. R. & LINDLEY, B. C. 1962 In *Advances in Magnetohydrodynamics*. London: Pergamon.

MAKIN, B. & KECK, J. C. 1963 *Phys. Rev. Letters*, **11**, 281.

MASSEY, H. S. W. & BURHOP, E. 1952 *Electronic and Ionic Impact Phenomena*. Oxford University Press.

MORSE, T. F. 1963 *Physics of Fluids*, **6**, 10.

NUE, H. 1962 *Symposium on magnetoplasmadynamic electrical power generation*. The Institution of Electrical Engineers, North Eastern Section, Session IV, Paper 25.

PETSCHEK, H. & BYRON, S. 1957. *Ann. Phys.* **1**, 270–315.

RUSSELL, C. R., BYRON, S. & BORTZ, P. I. 1963 Performance and Analysis of a Crossed-Field Accelerator. Paper read at *AIAA Electrical Propulsion Conference*, Colorado Springs.

SPITZER, L. 1956 *Physics of Fully Ionised Gases*. New York: Interscience.